

Harmonizing cubic Bézier curves

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1 Initial curvature of a cubic Bézier segment

A cubic Bézier segment can be described by:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = t^3(3\vec{Q} - \vec{P} + \vec{S} - 3\vec{R}) + 3t^2(\vec{P} - 2\vec{Q} + \vec{R}) + 3t(\vec{Q} - \vec{P}) + \vec{P}$$



The initial derivatives are then:

$$\begin{pmatrix} \dot{x}(0) \\ \dot{y}(0) \end{pmatrix} = 3(\underbrace{\vec{Q} - \vec{P}}_{=: \vec{v}}) \quad \begin{pmatrix} \ddot{x}(0) \\ \ddot{y}(0) \end{pmatrix} = 6(\underbrace{\vec{P} - 2\vec{Q} + \vec{R}}_{=: \vec{w}})$$

The signed curvature is calculated by $\frac{\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \times \begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix}}{|\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}|^3}$.

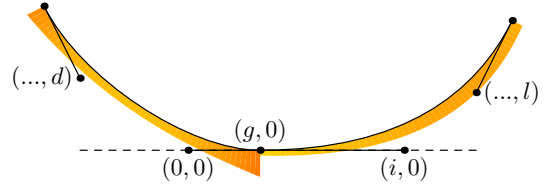
Using $v := |\vec{v}|$ we finally get the formula for the initial curvature (i.e. the curvature at time $t = 0$):

$$\frac{3\vec{v} \times 6\vec{w}}{(3v)^3} = \boxed{\frac{2}{3} \frac{\vec{v} \times \vec{w}}{v^3}}$$

The special case $|\begin{pmatrix} \dot{x}(0) \\ \dot{y}(0) \end{pmatrix}| = 0$ is not handled here. The curvature then would diverge to $\pm\infty$ (or be 0 if the cubic Bézier segment is a line segment).

2 The math of harmonization

Assume two adjoint cubic Bézier segments that have the same direction at their join and do not have zero-handles. Furthermore, assume the joining knot is not a point of inflection. By translation and rotation we can force one control point next to the joining knot to lie on the origin of the coordinate system and the joining knot tangent to lie on the x-axis. Additionally, the depicted coordinate d will be non-negative:

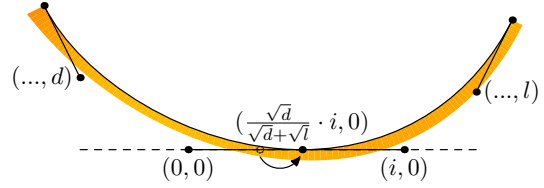


We want to choose g such that the curvature is continuous. So the curvature on both sides of $(g, 0)$ must be equal:

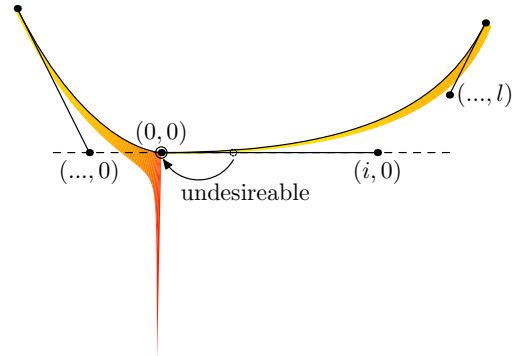
$$\frac{2d}{3g^2} = \frac{2l}{3(i-g)^2}$$

Solving for g and obeying the condition $0 \leq g \leq i$ we get

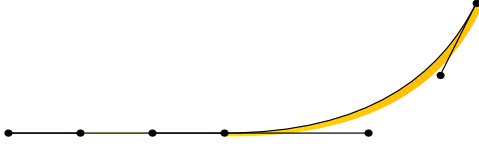
$$g = \frac{\sqrt{d}}{\sqrt{d} + \sqrt{l}} \cdot i.$$



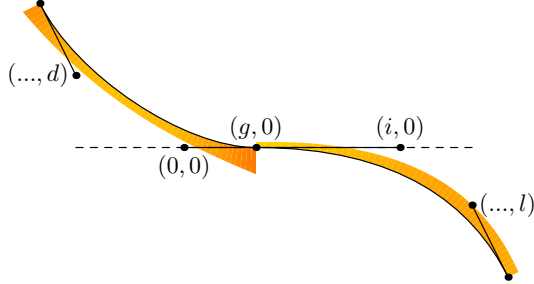
If either d or l is zero, $g = \frac{\sqrt{d}}{\sqrt{d} + \sqrt{l}} \cdot i$ becomes either 0 or i . That means the joining knot will become collocated with one of its control points, which generally should be avoided. One reason for this avoidance is that the curvature might become infinitely large:



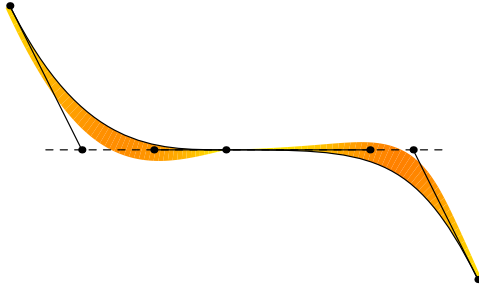
So, we will not alter the paths at all in the case of either d or l being zero. This case occurs also when a straight line goes over to a curve, which is quite frequent in type design:



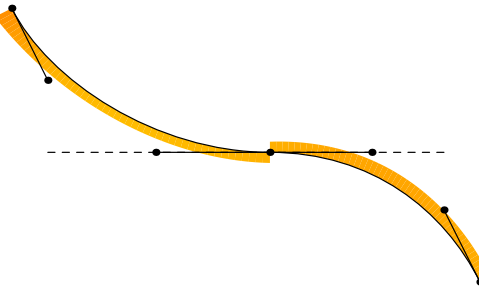
When the joining knot is a point of inflection, the curvatures $\frac{2d}{3g^2}$ and $\frac{2l}{3(i-g)^2}$ must have different signs.



Hence, a curvature-continuous solution forces $d = l = 0$, so all control points must lie on one line, as in:



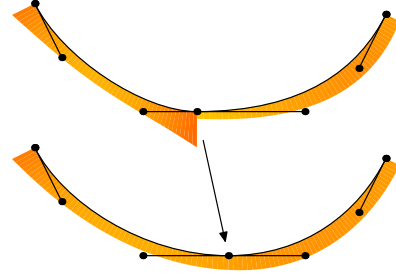
In this situation, one could also satisfy further conditions like the preservation of area. On the other hand, having all four control points on the same line as the two affected cubic Bézier segments is critical. Due to rounding errors, such a conversion may add new points of inflection. So, instead of this, we will only guarantee the *absolute value* of the curvature to be continuous in the case of points of inflection by moving the joining knot in the same manner as before, between its control points:



Finally, the solution of setting

$$g_{\text{new}} = \begin{cases} g_{\text{old}} & \text{if } d = 0 \text{ or } l = 0, \\ \frac{\sqrt{|d|}}{\sqrt{|d|} + \sqrt{|l|}} \cdot i & \text{else} \end{cases}$$

is chosen here and shall be the definition of *harmonization*. By harmonization the curvatures at the other segment ends will not change.



A mostly equivalent algorithm has been published in [1].

References

- [1] R.L. Roach. Curvature continuity of cubic Bezier curves in the solid modeling aerospace research tools design software. Interim report, NASA Langley Research Center, 1990. <https://ntrs.nasa.gov/citations/19900012238>